J. Math. Anal. Appl. 378 (2011) 749–763

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Comparison results for nonlinear degenerate Dirichlet and Neumann problems with general growth in the gradient ☆

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ARTICLE INFO

Article history:
Received 19 July 2010
Available online 2 February 2011
Submitted by A. Cianchi

Keywords:
Comparison results
Rearrangements
Degenerate
Nonlinear elliptic equation

ABSTRACT

This paper deals with both Dirichlet and Neumann problems for a class of nonlinear degenerate elliptic equations with general growth in the gradient. First, we give an existence result of a spherically symmetric solution to the "symmetrized" problems with data depending only on the radials. Second, we prove that the solutions of the original problems can be compared, under a rearrangement, with the solutions of the "symmetrized" problems.

1. Introduction and statement of main results

We first study a class of Dirichlet problem

$$\text{(P1)}\qquad \begin{cases} -\operatorname{div}\bigl(a(x,u,\nabla u)\bigr) = H(x,u,\nabla u) & \text{in } \Omega, \\ u \in W_0^{1,p}(\nu,\Omega)\cap L^\infty(\Omega), \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ $(N \geqslant 2)$, $a(x,s,\xi): \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ and $H(x,s,\xi): \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions such that

(i) $a(x,s,\xi)\cdot\xi \geqslant \nu(x)|\xi|^p$, for a.e. $x\in\Omega$, $\forall(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, $\nu(x)\geqslant 0$, $\nu\in L^h(\Omega)$, $h\geqslant 1$ and $\nu^{-1}\in L^t(\Omega)$, $1+\frac{1}{t}<p<N(1+\frac{1}{t})$, $t>\frac{N}{p}$;

(ii) $|a(x,s,\xi)|\nu^{-\frac{1}{p}} \leqslant c_1(k(x)+\nu^{\frac{1}{p'}}|s|^{p-1}+\nu^{\frac{1}{p'}}|\xi|^{p-1})$, for a.e. $x\in\Omega$, $\forall(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where $c_1>0$, $k(x)\geqslant 0$ and $k\in L^{p'}(\Omega)$, $p'=\frac{p}{p-1}$;

(iii) $|H(x,s,\xi)|\leqslant f(x)+\theta\nu^{\frac{q}{p}}|\xi|^q$, for a.e. $x\in\Omega$, $\forall(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where $\theta>0$, $p-1<q\leqslant p$, $f(x)\geqslant 0$ and $\frac{f}{\nu}\in L^r(\Omega)$, $\frac{1}{r}<\frac{p}{N}-\frac{1}{t}-\frac{1}{h}$.

☆ Project supported by NSFC (No. 10401009) and NCET of China (No. 060275).

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.01.072

The function u is said to be a solution to problem (P1), if $u \in W_0^{1,p}(\nu, \Omega) \cap L^\infty(\Omega)$ and

$$\int_\Omega a(x,u,\nabla u)\cdot \nabla\phi\,dx = \int_\Omega H(x,u,\nabla u)\phi\,dx, \quad \forall \phi \in W_0^{1,p}(\nu,\Omega)\cap L^\infty(\Omega). \tag{1.1}$$

Our first result is to exhibit a condition to ensure the existence of a spherically symmetric solution to the following "symmetrized" problem

$$\text{(P2)} \quad \begin{cases} -\operatorname{div}\big(\underline{\nu}\big(C_N|x|^N\big)|\nabla v|^{p-2}\nabla v\big) = f^\sharp(x) + \theta \underline{\nu}^{\frac{q}{p}}\big(C_N|x|^N\big)|\nabla v|^q \quad \text{in } \Omega^\sharp, \\ v \in W_0^{1,p}\big(\nu, \Omega^\sharp\big) \cap L^\infty\big(\Omega^\sharp\big), \end{cases}$$

where $\underline{\nu}^{-\frac{p'}{p}}$ is a pseudo-rearrangement of $\nu^{-\frac{p'}{p}}$ with respect to u (see Definition 2.4), $\Omega^\sharp$ is a ball centered at origin with the same measure as Ω, and $f^\sharp$ is the Schwarz symmetrization of f.

Theorem 1.1. *Set $\gamma = \frac{q}{p-1}$, $\gamma' = \frac{\gamma}{\gamma-1}$ and*

$$M_0 = \theta\beta^{\frac{\gamma}{pt}-1}\big(NC_N^{\frac{1}{N}}\big)^{-\gamma}|\Omega|^{\beta(1-\frac{\gamma}{pt})-\eta}\big\|\nu^{-1}\big\|_{L^t(\Omega)}^{\frac{\gamma}{p}},$$

where $\eta = 1-\frac{1}{r}-\frac{1}{h}$, $\beta = \gamma(\frac{1}{N}+\eta-1)(\frac{pt}{\gamma})'+1$. If

$$\left\|\frac{f}{\nu}\right\|_{L^r(\Omega)}\|\nu\|_{L^h(\Omega)} \leqslant \frac{1}{\gamma'}(\gamma M_0)^{\frac{1}{1-\gamma}}, \tag{1.2}$$

then there exists a unique solution to problem (P2) *such that $v(x) = v^\sharp(x)$. Moreover, $v \in C(\overline{\Omega^\sharp}) \cap W_0^{1,\widetilde{q}}(\underline{\nu}, \Omega^\sharp)$, where $p-1 \leqslant \widetilde{q} \leqslant (t+1)(p-1)$ if $1-\eta-\frac{1}{N} \geqslant 0$; $p-1 \leqslant \widetilde{q} < \frac{(t+1)(p-1)}{(1-\eta-\frac{1}{N})t+1}$ if $1-\eta-\frac{1}{N} < 0$, satisfying*

$$\|\nabla v\|_{L^p(\underline{\nu},\Omega^\sharp)}^p \leqslant \big(NC_N^{\frac{1}{N}}\big)^{-p'}(\gamma M_0)^{\frac{p'}{1-\gamma}}\left\|\frac{1}{\nu}\right\|_{L^t(\Omega)}^{\frac{1}{p-1}}\left(\frac{pt-t-1}{(\frac{1}{N}+\eta)pt-t-1}\right)^{1-\frac{1}{t(p-1)}} \times |\Omega|^{(\frac{1}{N}+\eta-1)p'+1-\frac{1}{t(p-1)}}$$

and

$$\|v\|_{L^\infty(\Omega^\sharp)} \leqslant \big(NC_N^{\frac{1}{N}}\big)^{-p'}(\gamma M_0)^{\frac{1}{(p-1)(1-\gamma)}}\left\|\frac{1}{\nu}\right\|_{L^t(\Omega)}\left(\frac{pt-t-1}{(\frac{p}{N}+\eta-1-\frac{1}{t})t}\right)^{1-\frac{1}{t(p-1)}} \times |\Omega|^{(\frac{p}{N}+\eta-1-\frac{1}{t})\frac{1}{p-1}}.$$

In the case $f(x) = f_0\nu(x)|x|^l$ with constants $l > -\frac{N}{r}$, $f_0 > 0$, we have the following nonexistence result for problem (P2).

Theorem 1.2. *Let $f(x) = f_0\nu(x)|x|^l$, where $l > -\frac{N}{r}$ and f_0 is a positive constant. Assume*

$$f_0^{\gamma-1} \geqslant \frac{\gamma^{\frac{\gamma}{\gamma-1}}(NC_N^{\frac{1}{N}})^\gamma|\Omega|^{\eta+\gamma}\||x|^l\|_{L^r(\Omega)}\|\nu\|_{L^h(\Omega)}}{\theta(\gamma-1)F_1(|\Omega|)}, \tag{1.3}$$

where $F_1 : [0, |\Omega|] \to \mathbb{R}$ is a function defined by

$$F_1(s) = \int_0^s \tau^{\frac{\gamma}{N}}\underline{\nu}^{-\frac{N}{p}}(\tau)A^\gamma(\tau)\,d\tau \quad \text{and} \quad A(s) = \int_0^s \big(\nu|x|^l\big)^\star(\tau)\,d\tau.$$

Then problem (P2) *has no spherically symmetric solution.*

Remark 1.1. By virtue of

$$F_1\big(|\Omega|\big) \leqslant \|\nu\|_{L^h(\Omega)}^\gamma\big\||x|^l\big\|_{L^r(\Omega)}^\gamma\big\|\nu^{-1}\big\|_{L^t(\Omega)}^{\frac{\gamma}{p}}|\Omega|^{(\eta+\frac{1}{N})\gamma+1-\frac{\gamma}{pt}},$$

it follows from (1.3) that

$$\begin{aligned}\| f_0|x|^l\|_{L^r(\Omega)}\|\nu\|_{L^h(\Omega)} &\geqslant \left(\frac{\gamma^{\frac{\gamma}{\gamma-1}}}{\gamma-1}\right)^{\frac{1}{\gamma-1}}\left[\theta\left(\gamma\left(\frac{1}{N}+\eta\right)\left(\frac{pt}{\gamma}\right)'+1\right)^{\frac{\gamma}{pt}-1}\right.\\ &\quad\left.\times\left(NC_N^{\frac{1}{N}}\right)^{-\gamma}|\Omega|^{\gamma(\frac{1}{N}+\eta-1)-\eta+1-\frac{\gamma}{pt}}\|\nu^{-1}\|_{L^t(\Omega)}^{\frac{\gamma}{p}}\right]^{\frac{1}{1-\gamma}}\\ &\geqslant \frac{1}{\gamma'}(\gamma M_0)^{\frac{1}{1-\gamma}}.\end{aligned}$$

Thus, Theorem 1.2 proposes an example to show that problem (P2) has no spherically symmetric solution without the assumption (1.2).

Actually, as $\nu\equiv 1$, several results similar to those explained above can be found in the literature. For the case $r=\infty$, under a smallness assumption on $\|f\|_{L^\infty(\Omega)}$, the existence of a spherically symmetric solution to problem (P2) was studied in [11,20] (for the case $q=p$), where the smallness assumption on $\|f\|_{L^\infty(\Omega)}$ depends not only on p, q, Ω, but also on the first eigenvalue of a nonlinear Dirichlet problem. However, neither regularity results with a priori estimates of solutions, nor any nonexistence results were contained in the two papers. Moreover, it seems that their approach cannot be extended to the case $f\notin L^\infty(\Omega)$. In [17], the case $f(x)=\widetilde{g}_0|x|^m$ with $q=p$ has been investigated for the existence and nonexistence of solutions, where $\widetilde{g}_0$ is a positive real number and $m>\max\{-p,-N\}$. As to nonexistence results, we can also refer to [23] for the case $f(x)\equiv f_0$ with $q=p$. To our knowledge, the existence and nonexistence of solutions to degenerate equations with general $f(x)$ (see (iii)) have been unsolved yet.

Now since the "symmetrized" problem (P2) admits a unique spherically symmetric solution under assumption (1.2), we have the following comparison results.

Theorem 1.3. *Assume that* (i)–(iii) *hold with f satisfying* (1.2)*. Let u be a solution to problem* (P1) *and v be the spherically symmetric solution to problem* (P2)*. Then*

$$u^\sharp(x)\leqslant v^\sharp(x),\quad x\in\Omega^\sharp \tag{1.4}$$

and

$$\int_\Omega \rho\big(\nu(x)|\nabla u|^p\big)\,dx\leqslant\int_{\Omega^\sharp}\rho\big(\underline{\nu}\big(\big(C_N|x|^N\big)\big)|\nabla v|^p\big)\,dx, \tag{1.5}$$

where ρ is a concave and nondecreasing function on $[0,+\infty)$.

As $\nu\equiv 1$, such comparison results have been obtained in [1] for the case $p=2$, in [22] for the case $f(x)\equiv f_0$ with $q=p$, and in [11,20] for the case $r=\infty$.

Next, we consider the following Neumann problem

$$(\mathrm{P3})\quad \begin{cases}-\operatorname{div}\big(a(x,u,\nabla u)\big)=H(x,u,\nabla u) & \text{in } \Omega,\\ a(x,u,\nabla u)\vec{n}=0 & \text{on } \partial\Omega,\end{cases}$$

where Ω is a bounded and connected open set with Lipschitz boundary, $a(x,s,\xi)$ and $H(x,s,\xi)$ satisfy the conditions (i), (ii) and

(iii′) $-f_2(x)-\theta_2\nu^{\frac{q}{p}}|\xi|^q\leqslant H(x,s,\xi)\leqslant f_1(x)+\theta_1\nu^{\frac{q}{p}}|\xi|^q$, for a.e. $x\in\Omega$, $\forall\xi\in\mathbb{R}^N$, where $\theta_i>0$, $f_i(x)\geqslant 0$, $\frac{f_i}{\nu}\in L^r(\Omega)$, $i=1,2$, q and r are the same as in (iii).

Remark 1.2. We assume (iii′) instead of (iii) in order to stress the different role of f_1, f_2, θ_1, θ_2 in bounding the positive and the negative part of the solution to problem (P3).

The function u is said to be a solution to problem (P3), if $u\in W^{1,p}(\nu,\Omega)\cap L^\infty(\Omega)$ and

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla\phi\,dx=\int_\Omega H(x,u,\nabla u)\phi\,dx,\quad \forall\phi\in W^{1,p}(\nu,\Omega)\cap L^\infty(\Omega). \tag{1.6}$$

Theorem 1.4. *Assume that* (i), (ii) *and* (iii′) *hold. Let u be a solution to problem* (P3)*. Set*

$$e=\inf\left\{t\in\mathbb{R}\colon \big|\{x\in\Omega\colon u(x)>t\}\big|\leqslant\frac{|\Omega|}{2}\right\},$$
$$u_1=(u-e)^+,\qquad u_2=(u-e)^-$$

and

$$M_{0i} = \theta_i \beta^{\frac{\gamma}{pt}-1} Q^{\gamma} \left(\frac{|\Omega|}{2}\right)^{\beta(1-\frac{\gamma}{pt})-\eta} \|\nu^{-1}\|_{L^t(\Omega)}^{\frac{\gamma}{p}}, \quad i = 1, 2,$$

where Q is the constant of (2.1) *and β the constant in Theorem* 1.1. *If*

$$\left\|\frac{f_i}{\nu}\right\|_{L^r(\Omega)} \|\nu\|_{L^h(\Omega)} \leqslant \frac{1}{\gamma'} (\gamma M_{0i})^{\frac{1}{1-\gamma}}, \quad i = 1, 2 \tag{1.7}$$

and v_i is the spherically symmetric solution to problem

$$\text{(P4i)} \quad \begin{cases} -\operatorname{div}\left(\underline{\nu_i}\left(C_N|x|^N\right)|\nabla v_i|^{p-2}\nabla v_i\right) = \alpha^p f_i^{\sharp}(x) + \theta_i \alpha^{p-q} \underline{\nu_i}^{\frac{q}{p}}\left(C_N|x|^N\right)|\nabla v_i|^q & \text{in } B, \\ v_i \in W_0^{1,p}(\nu, B) \cap L^{\infty}(B), \end{cases}$$

where $\alpha = Q N C_N^{\frac{1}{N}}$, B is the ball with center in the origin such that $|B| = \frac{|\Omega|}{2}$ and $\underline{\nu_i}^{-\frac{p'}{p}}$ is a pseudo-rearrangement of $\nu^{-\frac{p'}{p}}$ with respect to u_i, then

$$u_i^{\sharp}(x) \leqslant v_i^{\sharp}(x), \quad x \in B \tag{1.8}$$

and

$$\int_{\Omega} \rho\left(\nu(x)|\nabla u_i|^p\right) dx \leqslant \int_{B} \rho\left(\underline{\nu_i}\left(C_N|x|^N\right)|\nabla v_i|^p\right) dx, \quad i = 1, 2, \tag{1.9}$$

where ρ is a concave and nondecreasing function on $[0, +\infty)$.

As to comparison results for Neumann problems, only linear equations (see [5,10,18]) and the nonlinear case with $H(x, u, \nabla u) = f(x)$ (see [19]) were considered. Note that the corresponding "symmetrized" problem, being two Dirichlet problems composed, is no longer a Neumann problem.

Remark 1.3. Theorems 1.1, 1.3 and 1.4 provide a priori estimates on the solutions to problems (P1) and (P3). Thus, under an additional assumption

$$\text{(iv)} \quad \left[a(x, s, \xi_1) - a(x, s, \xi_2)\right] \cdot (\xi_1 - \xi_2) > 0 \quad \text{if } \xi_1 \neq \xi_2,$$

one can easily obtain the existence results for the two problems by using the well-known approximation techniques (see for example [6,7]).

There are three features of our work. First, the equations are degenerate. Second, a general assumption on f ((iii) or (iii′)) is allowed (in the previous results, either $f \in L^{\infty}(\Omega)$ or f has explicit forms). Finally, the Neumann problem with general growth in the gradient is discussed.

Our results are based on the study of a class of Volterra integral operator, which was first introduced by Pašić (see [22]). By proving a unique existence result for a nonlinear integral equation in suitable space, we obtain a unique spherically symmetric solution to problem (P2). Moreover, the solution possesses explicit integral representation. The comparison results are established by using a new type of comparison principle for Volterra integral operator (see Lemma 3.1).

This paper is organized as follows: we give some notations and preliminary results in Section 2, and then prove the main results of the paper in Section 3.

2. Notations and preliminary results

In this section, we recall some definitions and preliminary results.

Definition 2.1. (See [9,21].) Let E be a measurable subset of G, the De Giorgi perimeter of E with respect to G is defined as follows

$$P_G(E) = \sup\left\{\int_E \operatorname{div}\left(\psi(x)\right) dx\colon\ \psi = (\psi_1, \ldots, \psi_n) \in \left(C_0^1(G)\right)^N,\ \sup_{x \in G} |\psi| \leqslant 1\right\}.$$

From Fleming–Rishel Formula (see [14]), we have that if $u \in W_0^{1,1}(G)$, then

$$P_G\left(\left\{x \in G\colon\ \left|u(x)\right| > \tau\right\}\right) = -\frac{d}{d\tau} \int_{\{x \in G\colon\ |u(x)| > \tau\}} \left|\nabla u(x)\right| dx.$$

Definition 2.2. (See [13,14].) We say that G satisfies a relative isoperimetric inequality if there exists a constant Q depending only on G, such that for every measurable subset E of G,

$$\min\left\{|E|^{1-\frac{1}{N}}, |G\backslash E|^{1-\frac{1}{N}}\right\} \leqslant Q\, P_G(E). \tag{2.1}$$

For instance, if G is a bounded and connected open set with Lipschitz boundary, then G satisfies a relative isoperimetric inequality.

Especially, as $G = \mathbb{R}^N$, the classical isoperimetric inequality (see [9]) holds, i.e.

$$NC_N^{\frac{1}{N}}|E|^{1-\frac{1}{N}} \leqslant P_{\mathbb{R}^N}(E),$$

where E is a measurable bounded subset of $\mathbb{R}^N$ and C_N is the volume of the unit ball in $\mathbb{R}^N$.

Definition 2.3. Assume that u is a measurable function in Ω. Let

$$u^\star(s) = \inf\left\{\tau \geqslant 0\colon\ \mu(\tau) \leqslant s\right\}, \quad s \in \left[0, |\Omega|\right]$$

and

$$u_\star(s) = u^\star\left(|\Omega| - s\right), \quad s \in \left[0, |\Omega|\right]$$

denote the decreasing rearrangement and the increasing rearrangement of u respectively, where $\mu(\tau) = |\{x \in \Omega\colon\ |u| > \tau\}|$ is the distribution function of u.

Let

$$u^\sharp(x) = u^\star\left(C_N|x|^N\right), \quad x \in \Omega^\sharp$$

and

$$u_\sharp(x) = u_\star\left(C_N|x|^N\right), \quad x \in \Omega^\sharp$$

denote the decreasing spherically symmetric rearrangement (Schwarz symmetrization) and the increasing spherically symmetric rearrangement of u respectively, where $\Omega^\sharp$ is the ball centered at the original with the same measure as Ω.

For an exhaustive treatment of the properties of rearrangements we refer to [4,8,15,16,21,25]. Here we just quote the following properties.

(a) If u and v are measurable functions, the Hardy–Littlewood inequality

$$\int_0^{|\Omega|} u_\star(s)v^\star(s)\,ds = \int_{\Omega^\sharp} u_\sharp(x)v^\sharp(x)\,dx \leqslant \int_\Omega \left|u(x)v(x)\right|dx$$
$$\leqslant \int_{\Omega^\sharp} u^\sharp(x)v^\sharp(x)\,dx = \int_0^{|\Omega|} u^\star(s)v^\star(s)\,ds$$

holds.

(b) The L^p-norm is invariant under Schwarz symmetrization,

$$\|u\|_{L^p(\Omega)} = \left\|u^\sharp\right\|_{L^p(\Omega^\sharp)} = \left\|u^\star\right\|_{L^p(0,|\Omega|)}, \quad 1 \leqslant p \leqslant +\infty.$$

Next, let's introduce the following notion of pseudo-rearrangement first introduced in [2].

Definition 2.4. Let $u : \Omega \to \mathbb{R}$ be a measurable function, $f \in L^1(\Omega)$, $f(x) \geqslant 0$ and $\Omega^\star = [0, |\Omega|]$. We say that a function $\underline{f}(\sigma) : \Omega^\star \to \mathbb{R}$ is a pseudo-rearrangement of f with respect to u if there exists a family $\{E(\sigma)\}_{\sigma\in\Omega^\star}$ of measurable subsets of Ω such that

$$\gamma_n\left(E(\sigma)\right) = \sigma,$$
$$\sigma_1 \leqslant \sigma_2 \quad\Rightarrow\quad E(\sigma_1) \subseteq E(\sigma_2),$$
$$E(\sigma) = \left\{x \in \Omega\colon\ \left|u(x)\right| > u^\star(\sigma)\right\}, \quad \text{if}\ \sigma = \mu(\tau)$$

and

$$\underline{f}(\sigma) = \frac{d}{d\sigma} \int\limits_{E(\sigma)} f(x)\,dx, \quad \text{a.e. } \sigma \in \Omega^\star.$$

For general results about the properties of pseudo-rearrangement we can refer to [3,12,24]. We just mention that

Proposition 2.1. *If $f \in L^p(\Omega)$ with $p \geqslant 1$, then $\underline{f} \in L^p(0, |\Omega|)$ and*

$$\|\underline{f}\|_{L^p(0,|\Omega|)} \leqslant \|f\|_{L^p(\Omega)}.$$

3. Proof of the main results

We first give some useful lemmas to the proofs of Theorems 1.1–1.4.

Assume that $g(s,\xi) : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. We consider the following Volterra integral operator (see [17,22,26])

$$K : D(K) \subseteq C\big([0,T]\big) \to C\big([0,T]\big), \quad K\psi(t) = \int\limits_0^t g\big(\tau, \psi(\tau)\big)\,d\tau, \quad \forall \psi \in D(K). \tag{3.1}$$

The following definition is some different from that in [17]. Ours is more concise and readily comprehensible.

Definition 3.1. We say that the operator K has property (m) if for all $\psi_1, \psi_2 \in D(K)$ and $a \in [0,T)$, there exist three constants $\eta > 0$, $b \in (a,T]$ and $m(a,b,\psi_1,\psi_2) \in [0,1)$ such that for any $\sigma \in (a,b]$,

$$\big\|g\big(\cdot,\psi_1(\cdot)\big) - g\big(\cdot,\psi_2(\cdot)\big)\big\|_{L^1(a,\sigma)} \leqslant \sigma^\eta m(a,b,\psi_1,\psi_2) \left\| \frac{\psi_1 - \psi_2}{s^\eta} \right\|_{L^\infty(a,\sigma)}. \tag{3.2}$$

Remark 3.1. If we can obtain (3.2) for all $b \in (a,T]$ and the following property on $m(a,b,\psi_1,\psi_2)$ is satisfied

$$\lim_{b\to a^+} m(a,b,\psi_1,\psi_2) = 0,$$

then the operator K has property (m).

The following comparison principle has been obtained in [17]. However, by virtue of the different definition of property (m) here, we give a simple proof.

Lemma 3.1. *Let K be a Volterra integral operator given by* (3.1) *satisfying property* (m) *and $R(K) \subseteq D(K)$, where $R(K)$ is the range of K. If $g(\sigma,\cdot)$ is nondecreasing for a.e. $\sigma \in [0,T]$, then for $\forall u, v \in D(K)$ satisfying $u \leqslant Ku$, $v \geqslant Kv$, we have*

$$u \leqslant v. \tag{3.3}$$

In particular, the equation $z = Kz$ possesses at most one solution in $D(K)$.

Proof. We argue by contradiction. If (3.3) does not hold, there must exist $a \in [0,T]$ and $b_1 \in (a,T]$ such that $u(a) = v(a)$ and $u(\sigma) > v(\sigma)$ for all $\sigma \in (a,b_1]$. Set $b_2 = \min\{b_1, b\}$, where b is the constant in Definition 3.1.

Due to property (m), it follows that for $\forall \sigma \in (a,b_2]$,

$$\begin{aligned} \big|u(\sigma) - v(\sigma)\big| = u(\sigma) - v(\sigma) &= \int\limits_a^\sigma \big(g\big(\tau,u(\tau)\big) - g\big(\tau,v(\tau)\big)\big)\,d\tau \\ &\leqslant \big\|g\big(\cdot,u(\cdot)\big) - g\big(\cdot,v(\cdot)\big)\big\|_{L^1(a,\sigma)} \leqslant \sigma^\eta m(a,b,u,v) \left\| \frac{u-v}{s^\eta} \right\|_{L^\infty(a,\sigma)}. \end{aligned}$$

Taking the maximum over $\sigma \in (a,b_2]$, we have

$$\left\| \frac{u-v}{s^\eta} \right\|_{L^\infty(a,b_2)} \leqslant m(a,b,u,v) \left\| \frac{u-v}{s^\eta} \right\|_{L^\infty(a,b_2)}.$$

Consequently, $u = v$ on $(a,b_2]$, which is a contradiction. Thus, (3.3) holds, and then the unique claim easily follows. □

Now, more precisely, let

$$K\psi(s)=\int_0^s f^\star(\tau)+\theta\big(NC_N^{\frac{1}{N}}\tau^{1-\frac{1}{N}}\big)^{-\gamma}\underline{\nu}^{-\frac{\gamma}{p}}(\tau)\psi^\gamma(\tau)\,d\tau,\quad s\in\big[0,|\Omega|\big],$$

$$D_1(K)=\big\{\psi\in C\big(\big[0,|\Omega|\big]\big):\ M\geqslant 0,\ 0\leqslant\psi(s)\leqslant Ms^\eta\big\},$$

$$D_2(K)=\big\{\psi\in C\big(\big[0,|\Omega|\big]\big):\ \exists M_\psi\geqslant 0,\ 0\leqslant\psi(s)\leqslant M_\psi s^\eta\big\}$$

and

$$R_i(K)\text{ is the range of }K\text{ on }D_i(K),\quad i=1,2,$$

where M in $D_1(K)$ is independent of ψ while M_ψ in $D_2(K)$ may change with the choice of ψ.

Lemma 3.2. *Let $M=(\gamma M_0)^{\frac{1}{1-\gamma}}$ in $D_1(K)$. If (1.2) holds, then there exists $w\in D_1(K)$ such that $w=Kw$. Furthermore, w is unique in $D_2(K)$.*

Proof. Step 1. We first claim that $R_1(K)\subseteq D_1(K)$.

In fact, by Hölder inequality and Proposition 2.1, for $\forall\psi\in D_1(K)$ and $s\in[0,|\Omega|]$,

$$\begin{aligned}K\psi(s)&=\int_0^s f^\star(\tau)+\theta\big(NC_N^{\frac{1}{N}}\tau^{1-\frac{1}{N}}\big)^{-\gamma}\underline{\nu}^{-\frac{\gamma}{p}}\psi^\gamma(\tau)\,d\tau\\
&\leqslant\left\|\frac{f^\star}{\nu_\star}\right\|_{L^r(\Omega)}\|\nu_\star\|_{L^h(\Omega)}s^\eta+\theta\big(NC_N^{\frac{1}{N}}\big)^{-\gamma}M^\gamma\big\|\underline{\nu}^{-1}\big\|^{\frac{\gamma}{p}}_{L^t(0,|\Omega|)}\times\left(\int_0^s\tau^{\gamma(\frac{1}{N}-1+\eta)(\frac{pt}{\gamma})'}\,d\tau\right)^{1-\frac{\gamma}{pt}}\\
&=\left\|\frac{f}{\nu}\right\|_{L^r(\Omega)}\|\nu\|_{L^h(\Omega)}s^\eta+\theta\big(NC_N^{\frac{1}{N}}\big)^{-\gamma}M^\gamma\big\|\underline{\nu}^{-1}\big\|^{\frac{\gamma}{p}}_{L^t(0,|\Omega|)}\left(\frac{s^\beta}{\beta}\right)^{1-\frac{\gamma}{pt}}\\
&\leqslant\left\|\frac{f}{\nu}\right\|_{L^r(\Omega)}\|\nu\|_{L^h(\Omega)}s^\eta+\theta\big(NC_N^{\frac{1}{N}}\big)^{-\gamma}M^\gamma\big\|\nu^{-1}\big\|^{\frac{\gamma}{p}}_{L^t(\Omega)}\left(\frac{s^\beta}{\beta}\right)^{1-\frac{\gamma}{pt}}.\end{aligned}$$

By virtue of $\frac{1}{r}<\frac{p}{N}-\frac{1}{t}-\frac{1}{h}$, we have $\beta(1-\frac{\gamma}{pt})>\eta$. Therefore,

$$K\psi(s)\leqslant\left(\left\|\frac{f}{\nu}\right\|_{L^r(\Omega)}\|\nu\|_{L^h(\Omega)}+M_0M^\gamma\right)s^\eta,\tag{3.4}$$

where $M_0=\theta\beta^{\frac{\gamma}{pt}-1}(NC_N^{\frac{1}{N}})^{-\gamma}|\Omega|^{\beta(1-\frac{\gamma}{pt})-\eta}\|\nu^{-1}\|^{\frac{\gamma}{p}}_{L^t(\Omega)}$.

Since $M=(\gamma M_0)^{\frac{1}{1-\gamma}}$ and (1.2) holds, we obtain

$$\left\|\frac{f}{\nu}\right\|_{L^r(\Omega)}\|\nu\|_{L^h(\Omega)}+M_0M^\gamma\leqslant M,$$

which implies $K\psi(s)\leqslant M\psi(s)$, $s\in[0,|\Omega|]$. Thus $R_1(K)\subseteq D_1(K)$.

Step 2. We prove that the operator K is compact with respect to the uniform topology of $C([0,|\Omega|])$.

Let's first show the equicontinuity of $R_1(K)$ in $C([0,|\Omega|])$. Taking any $K\psi\in R_1(K)$, for $0\leqslant a<b\leqslant|\Omega|$, we have

$$\begin{aligned}\big|K\psi(b)-K\psi(a)\big|&\leqslant\int_a^b f^\star(\tau)+\theta\big(NC_N^{\frac{1}{N}}\tau^{1-\frac{1}{N}}\big)^{-\gamma}\underline{\nu}^{-\frac{\gamma}{p}}\psi^\gamma(\tau)\,d\tau\\
&\leqslant\left\|\frac{f}{\nu}\right\|_{L^r(\Omega)}\|\nu\|_{L^h(\Omega)}(b-a)^\eta+M_0M^\gamma\big(b^\beta-a^\beta\big)^{\frac{\eta}{\beta}}.\end{aligned}$$

Therefore, $|K\psi(b)-K\psi(a)|$ tends to zero uniformly as $b\to a$, i.e. $R_1(K)$ is equicontinuous.

Also, the family of functions from $R_1(K)$ is uniformly bounded (see (3.4)), i.e.

$$0\leqslant K\psi\leqslant\left(\left\|\frac{f}{\nu}\right\|_{L^r(\Omega)}\|\nu\|_{L^h(\Omega)}+M_0M^\gamma\right)|\Omega|^\eta.$$

Ascoli–Arzela theorem implies that $R_1(K)$ is relatively compact in $C([0,|\Omega|])$. Hence the operator K is compact.

Note that the domain $D_1(K)$ is bounded, closed and convex. By the facts that $R_1(K) \subseteq D_1(K)$ and K is compact, Schauder's fixed point theorem shows that there exists at least one solution $w \in D_1(K)$ satisfying $w = Kw$.

Step 3. We assert that the solution w obtained in Step 2 is unique in $D_2(K)$.

Firstly, K has property (m) in $D_2(K)$.

Indeed, for $\forall \psi_1, \psi_2 \in D_2(K)$, $0 \leqslant a < b \leqslant |\Omega|$ and $a < \sigma \leqslant b$,

$$\begin{aligned}
&\big\| g\big(\cdot, \psi_1(\cdot)\big) - g\big(\cdot, \psi_2(\cdot)\big)\big\|_{L^1(a,\sigma)} \\
&= \int_a^\sigma \theta \big(NC_N^{\frac{1}{N}} \tau^{1-\frac{1}{N}}\big)^{-\gamma} \underline{\nu}^{-\frac{\gamma}{p}}(\tau) \big|\psi_1^\gamma(\tau) - \psi_2^\gamma(\tau)\big|\, d\tau \\
&\leqslant \theta\gamma \big(NC_N^{\frac{1}{N}}\big)^{-\gamma} \int_a^\sigma \tau^{(\frac{1}{N}-1)\gamma} \underline{\nu}^{-\frac{\gamma}{p}}(\tau) \max\big\{\psi_1^{\gamma-1}(\tau), \psi_2^{\gamma-1}(\tau)\big\} \big|\psi_1(\tau) - \psi_2(\tau)\big|\, d\tau \\
&\leqslant \theta\gamma \big(NC_N^{\frac{1}{N}}\big)^{-\gamma} \max\{M_{\psi_1}, M_{\psi_2}\}^{\gamma-1} \int_a^\sigma \tau^{(\frac{1}{N}-1)\gamma+(\gamma-1)\eta} \underline{\nu}^{-\frac{\gamma}{p}}(\tau) \big|\psi_1(\tau) - \psi_2(\tau)\big|\, d\tau \\
&\leqslant \theta\gamma\beta^{\frac{\gamma}{pt}-1} \big(NC_N^{\frac{1}{N}}\big)^{-\gamma} \widetilde{M}^{\gamma-1} \big\|\underline{\nu}^{-1}\big\|_{L^t(0,|\Omega|)}^{\frac{\gamma}{p}} \big(\sigma^\beta - a^\beta\big)^{1-\frac{\gamma}{pt}} \left\| \frac{\psi_1 - \psi_2}{\tau^\eta} \right\|_{L^\infty(a,\sigma)} \\
&\leqslant \theta\gamma\beta^{\frac{\gamma}{pt}-1} \big(NC_N^{\frac{1}{N}}\big)^{-\gamma} \widetilde{M}^{\gamma-1} \big\|\nu^{-1}\big\|_{L^t(\Omega)}^{\frac{\gamma}{p}} \frac{(b^\beta - a^\beta)^{1-\frac{\gamma}{pt}}}{b^\eta} \left\| \frac{\psi_1 - \psi_2}{\tau^\eta} \right\|_{L^\infty(a,\sigma)} \sigma^\eta \\
&= m(a, b, \psi_1, \psi_2) \left\| \frac{\psi_1 - \psi_2}{\tau^\eta} \right\|_{L^\infty(a,\sigma)} \sigma^\eta,
\end{aligned}$$

where

$$\widetilde{M} = \max\{M_{\psi_1}, M_{\psi_2}\}$$

and

$$m(a, b, \psi_1, \psi_2) = \theta\gamma\beta^{\frac{\gamma}{pt}-1} \big(NC_N^{\frac{1}{N}}\big)^{-\gamma} \widetilde{M}^{\gamma-1} \big\|\nu^{-1}\big\|_{L^t(\Omega)}^{\frac{\gamma}{p}} \frac{(b^\beta - a^\beta)^{1-\frac{\gamma}{pt}}}{b^\eta}.$$

Observing $\beta(1 - \frac{\gamma}{pt}) - \eta > 0$, it follows

$$\lim_{b \to a^+} m(a, b, \psi_1, \psi_2) = 0.$$

Thus, from Remark 3.1, K satisfies property (m).

On the other hand, it is easy to prove that $R_2(K) \subseteq D_2(K)$. Then the uniqueness of w follows from Lemma 3.1. $\square$

The following lemma is also needed.

Lemma 3.3. *Let u be a solution to problem* (P1) *and v be the solution to problem* (P2) *such that $v = v^\sharp$. Then we have*

$$\big(-u^{\star'}(s)\big)^{p-1} \big(NC_N^{\frac{1}{N}} s^{1-\frac{1}{N}}\big)^p \underline{\nu}(s) \leqslant \int_0^s f^\star(\tau) \exp\left[\theta \int_\tau^s \big(NC_N^{\frac{1}{N}} \sigma^{1-\frac{1}{N}}\big)^{q-p} \underline{\nu}^{\frac{q}{p}-1}(\sigma) \big(-u^{\star'}(\sigma)\big)^{q-p+1} d\sigma\right] d\tau \tag{3.5}$$

and

$$\begin{aligned}
&\big(-v^{\star'}(s)\big)^{p-1} \big(NC_N^{\frac{1}{N}} s^{1-\frac{1}{N}}\big)^p \underline{\nu}(s) \\
&= \int_0^s f^\star(\tau) \exp\left[\theta \int_\tau^s \big(NC_N^{\frac{1}{N}} \sigma^{1-\frac{1}{N}}\big)^{q-p} \underline{\nu}^{\frac{q}{p}-1}(\sigma) \big(-v^{\star'}(\sigma)\big)^{q-p+1} d\sigma\right] d\tau, \quad \text{a.e. in } \big(0, |\Omega|\big).
\end{aligned} \tag{3.6}$$

A detailed proof of Lemma 3.3 is not supplied here since it follows the same lines as in [11].

Proof of Theorem 1.1. Step 1. Let

$$v(x)=V\big(C_N|x|^N\big)=\int_{C_N|x|^N}^{|\Omega|}\big(NC_N^{\frac{1}{N}}\tau^{1-\frac{1}{N}}\big)^{-p'}w^{\frac{p'}{p}}(\tau)\underline{\nu}^{-\frac{p'}{p}}(\tau)\,d\tau,\quad x\in\Omega^\sharp,\tag{3.7}$$

where w is the unique solution obtained in Lemma 3.2.

Firstly, since $w\in D_1(K)$, for $\widetilde{q}\geqslant p-1$,

$$\begin{aligned}\int_{\Omega^\sharp}\underline{\nu}\big(C_N|x|^N\big)|\nabla v|^{\widetilde{q}}\,dx&=\int_0^{|\Omega|}\underline{\nu}(s)\big(NC_N^{\frac{1}{N}}\big)^{-\frac{\widetilde{q}}{p-1}}\big(w(s)s^{\frac{1}{N}-1}\underline{\nu}^{-1}(s)\big)^{\frac{\widetilde{q}}{p-1}}\,ds\\&\leqslant\big(NC_N^{\frac{1}{N}}\big)^{-\frac{\widetilde{q}}{p-1}}M^{\frac{\widetilde{q}}{p-1}}\int_0^{|\Omega|}\big(s^{\frac{1}{N}-1+\eta}\big)^{\frac{\widetilde{q}}{p-1}}\big(\underline{\nu}^{-1}(s)\big)^{\frac{\widetilde{q}}{p-1}-1}\,ds\\&\leqslant\big(NC_N^{\frac{1}{N}}\big)^{-\frac{\widetilde{q}}{p-1}}M^{\frac{\widetilde{q}}{p-1}}\left\|\frac{1}{\underline{\nu}}\right\|_{L^t(0,|\Omega|)}^{\frac{\widetilde{q}}{p-1}-1}\left(\int_0^{|\Omega|}s^{(\frac{1}{N}-1+\eta)\frac{\widetilde{q}}{p-1}(\frac{t(p-1)}{\widetilde{q}-p+1})'}\,ds\right)^{1-\frac{\widetilde{q}-p+1}{t(p-1)}}\\&\leqslant\big(NC_N^{\frac{1}{N}}\big)^{-\frac{\widetilde{q}}{p-1}}M^{\frac{\widetilde{q}}{p-1}}\left\|\frac{1}{\nu}\right\|_{L^t(\Omega)}^{\frac{\widetilde{q}}{p-1}-1}\left(\int_0^{|\Omega|}s^{(\frac{1}{N}-1+\eta)\frac{\widetilde{q}}{p-1}(\frac{t(p-1)}{\widetilde{q}-p+1})'}\,ds\right)^{1-\frac{\widetilde{q}-p+1}{t(p-1)}}.\end{aligned}$$

Thus, as $\frac{1}{N}+\eta-1\geqslant 0$, $v\in W_0^{1,\widetilde{q}}(\underline{\nu},\Omega^\sharp)$ with $p-1\leqslant\widetilde{q}\leqslant(t+1)(p-1)$; as $\frac{1}{N}+\eta-1<0$, $v\in W_0^{1,\widetilde{q}}(\underline{\nu},\Omega^\sharp)$ with $p-1\leqslant\widetilde{q}<\frac{(t+1)(p-1)}{(1-\eta-\frac{1}{N})t+1}$.

In particular, observing that $\frac{1}{r}<\frac{p}{N}-\frac{1}{t}-\frac{1}{h}$ and $1+\frac{1}{t}<p<N(1+\frac{1}{t})$, we have that $p<(t+1)(p-1)$ and as $\frac{1}{N}-1+\eta<0$, $p<\frac{(t+1)(p-1)}{(1-\eta-\frac{1}{N})t+1}$. Hence $v\in W_0^{1,p}(\underline{\nu},\Omega^\sharp)$ and

$$\begin{aligned}\int_{\Omega^\sharp}\underline{\nu}\big(C_N|x|^N\big)|\nabla v|^p\,dx&\leqslant\big(NC_N^{\frac{1}{N}}\big)^{-p'}M^{p'}\left\|\frac{1}{\nu}\right\|_{L^t(\Omega)}^{\frac{1}{p-1}}\left(\int_0^{|\Omega|}s^{(\frac{1}{N}+\eta-1)\frac{pt}{pt-t-1}}\,ds\right)^{1-\frac{1}{t(p-1)}}\\&=\big(NC_N^{\frac{1}{N}}\big)^{-p'}M^{p'}\left\|\frac{1}{\nu}\right\|_{L^t(\Omega)}^{\frac{1}{p-1}}\left(\frac{pt-t-1}{(\frac{1}{N}+\eta)pt-t-1}\right)^{1-\frac{1}{t(p-1)}}\times|\Omega|^{(\frac{1}{N}+\eta-1)p'+1-\frac{1}{t(p-1)}}.\end{aligned}$$

Moreover,

$$\begin{aligned}\|v\|_{L^\infty(\Omega)}=V(0)&=\int_0^{|\Omega|}\big(NC_N^{\frac{1}{N}}\tau^{1-\frac{1}{N}}\big)^{-p'}w^{\frac{p'}{p}}(\tau)\underline{\nu}^{-\frac{p'}{p}}(\tau)\,d\tau\\&\leqslant\big(NC_N^{\frac{1}{N}}\big)^{-p'}M^{\frac{1}{p-1}}\int_0^{|\Omega|}\tau^{(\frac{1}{N}-1)p'+\frac{\eta}{p-1}}\underline{\nu}^{-\frac{p'}{p}}(\tau)\,d\tau\\&\leqslant\big(NC_N^{\frac{1}{N}}\big)^{-p'}M^{\frac{1}{p-1}}\left\|\frac{1}{\underline{\nu}}\right\|_{L^t(\Omega)}\left(\int_0^{|\Omega|}\tau^{[(\frac{1}{N}-1)p'+\frac{\eta}{p-1}][(p-1)t]'}\,d\tau\right)^{1-\frac{1}{t(p-1)}}\\&\leqslant\big(NC_N^{\frac{1}{N}}\big)^{-p'}M^{\frac{1}{p-1}}\left\|\frac{1}{\nu}\right\|_{L^t(\Omega)}\left(\int_0^{|\Omega|}\tau^{[(\frac{1}{N}-1)p'+\frac{\eta}{p-1}][(p-1)t]'}\,d\tau\right)^{1-\frac{1}{t(p-1)}}\\&=\big(NC_N^{\frac{1}{N}}\big)^{-p'}M^{\frac{1}{p-1}}\left\|\frac{1}{\nu}\right\|_{L^t(\Omega)}\left(\frac{pt-t-1}{(\frac{p}{N}+\eta-1-\frac{1}{t})t}\right)^{1-\frac{1}{t(p-1)}}\times|\Omega|^{(\frac{p}{N}+\eta-1-\frac{1}{t})\frac{1}{p-1}}.\end{aligned}$$

Also, $(NC_N^{\frac{1}{N}}\tau^{1-\frac{1}{N}})^{-p'}w^{\frac{p'}{p}}\underline{\nu}^{-\frac{p'}{p}}\in L^1(0,|\Omega|)$. In this way, $V\in C[0,|\Omega|]$ and then $v\in C(\overline{\Omega^\sharp})$.

Now, we prove that v is a solution to problem (P2).

In fact, by virtue of (3.7) and the fact that w is the unique solution of $Kw = w$ in $D_1(K)$, for $\forall\psi \in W_0^{1,p}(\underline{\nu}, \Omega^\sharp) \cap L^\infty(\Omega^\sharp)$,

$$\begin{aligned}\int_{\Omega^\sharp} \underline{\nu}\big(C_N|x|^N\big)|\nabla v|^{p-2}\nabla v \cdot \nabla\psi\, dx &= -\int_{\Omega^\sharp} \underline{\nu}\big(C_N|x|^N\big)\big(-V'\big(C_N|x|^N\big)\big)^{p-1}\big(NC_N|x|^{N-1}\big)^{p-1}\frac{x}{|x|}\cdot\nabla\psi\, dx \\ &= -\int_{\Omega^\sharp} w\big(C_N|x|^N\big)\big(NC_N|x|^{N-1}\big)^{-1}\frac{x}{|x|}\cdot\nabla\psi\, dx \\ &= \int_{\Omega^\sharp} (NC_N)^{-1} D_i w\frac{x_i}{|x|^N}\psi + (NC_N)^{-1} w\psi D_i\left(\frac{x_i}{|x|^N}\right) dx \\ &= \int_{\Omega^\sharp} \big(f^\sharp(x) + \theta\underline{\nu}^{\frac{q}{p}}\big(C_N|x|^N\big)|\nabla v|^q\big)\psi\, dx.\end{aligned}$$

Step 2. v is the unique spherically symmetric solution to problem (P2).

Indeed, let $\widetilde{v} \in W_0^{1,p}(\underline{\nu}, \Omega^\sharp) \cap L^\infty(\Omega^\sharp)$ be another spherically symmetric solution to problem (P2) and

$$\widetilde{w}(s) = \int_0^s f^\star(\tau)\exp\left[\theta\int_\tau^s \big(NC_N^{\frac{1}{N}}\sigma^{1-\frac{1}{N}}\big)^{q-p}\underline{\nu}^{\frac{q}{p}-1}(\sigma)\big(-\widetilde{v}^{\star\prime}(\sigma)\big)^{q-p+1}d\sigma\right]d\tau. \tag{3.8}$$

Thus, we obtain from (3.6) that

$$-\widetilde{v}^{\star\prime}(s) = \big(NC_N^{\frac{1}{N}}s^{1-\frac{1}{N}}\big)^{-p'}\widetilde{w}^{\frac{1}{p-1}}(s)\underline{\nu}^{-\frac{1}{p-1}}(s), \quad \text{a.e. } s \in \big(0, |\Omega|\big). \tag{3.9}$$

According to (3.8) and (3.9), for a.e. $s \in (0, |\Omega|)$,

$$\begin{aligned}\widetilde{w}'(s) &= f^\star(s) + \theta\big(NC_N^{\frac{1}{N}}s^{1-\frac{1}{N}}\big)^{q-p}\big(-\widetilde{v}^{\star\prime}(s)\big)^{q-p+1}\underline{\nu}^{\frac{q}{p}-1}(s)\widetilde{w}(s) \\ &= f^\star(s) + \theta\big(NC_N^{\frac{1}{N}}s^{1-\frac{1}{N}}\big)^{-\gamma}\underline{\nu}^{-\frac{\gamma}{p}}(s)\widetilde{w}^\gamma(s).\end{aligned}$$

Noting $\widetilde{w}(0) = 0$, we obtain

$$\widetilde{w}(s) = \int_0^s f^\star(\tau) + \theta\big(NC_N^{\frac{1}{N}}\tau^{1-\frac{1}{N}}\big)^{-\gamma}\underline{\nu}^{-\frac{\gamma}{p}}(\tau)\widetilde{w}^\gamma(\tau)\, d\tau. \tag{3.10}$$

Furthermore, by Hölder inequality and the fact $t > \frac{N}{p}$, (3.8) implies

$$\begin{aligned}\widetilde{w}(s) &\leqslant \int_0^s f^\star(\tau)\exp\left[\theta\big(NC_N^{\frac{1}{N}}\big)^{q-p}\left(\int_\tau^s \sigma^{(\frac{1}{N}-1)(pt)'}d\sigma\right)^{1-\frac{1}{pt}} \times \left(-\int_\tau^s \widetilde{v}^{\star\prime}(\sigma)d\sigma\right)^{q-p+1}\left\|\frac{1}{\underline{\nu}}\right\|_{L^t(0,|\Omega|)}^{\frac{p-q}{pt}}\right]d\tau \\ &\leqslant \int_0^s f^\star(\tau)\, d\tau\exp\left[\theta\left(\frac{pt-N}{(pt-1)N}\right)^{\frac{1}{pt}-1}\big(NC_N^{\frac{1}{N}}\big)^{q-p} \times \|\widetilde{v}\|_{L^\infty(\Omega^\sharp)}^{q-p+1}|\Omega|^{\frac{1}{N}-\frac{1}{pt}}\left\|\frac{1}{\nu}\right\|_{L^t(\Omega)}^{\frac{p-q}{pt}}\right] \\ &\leqslant \left\|\frac{f}{\nu}\right\|_{L^r(\Omega)}\|\nu\|_{L^h(\Omega)}\exp\left[\theta\left(\frac{pt-N}{(pt-1)N}\right)^{\frac{1}{pt}-1}\big(NC_N^{\frac{1}{N}}\big)^{q-p} \times \|\widetilde{v}\|_{L^\infty(\Omega^\sharp)}^{q-p+1}|\Omega|^{\frac{1}{N}-\frac{1}{pt}}\left\|\frac{1}{\nu}\right\|_{L^t(\Omega)}^{\frac{p-q}{pt}}\right]s^\eta,\end{aligned}$$

which shows that $\widetilde{w} \in D_2(K)$.

Then, we can conclude from (3.10) that $\widetilde{w} = w$. By (3.9) and the fact that $\widetilde{v}^\star(|\Omega|) = 0$, we get $\widetilde{v} = v$. Thus the uniqueness is proved. □

Remark 3.2. From the above proof, we see that $v^\star$ and w can be expressed by each other via the following equations:

$$v^\star(s) = \int_s^{|\Omega|} \big(NC_N^{\frac{1}{N}}\tau^{1-\frac{1}{N}}\big)^{-p'}\underline{\nu}^{-\frac{p'}{p}}(\tau)w^{\frac{p'}{p}}(\tau)\, d\tau, \tag{3.11}$$

and

$$w(s)=\int_0^s f^\star(\tau)\exp\left[\theta\int_\tau^s \left(NC_N^{\frac{1}{N}}\sigma^{1-\frac{1}{N}}\right)^{-(p-q)}\underline{v}^{\frac{q}{p}-1}(\sigma)\times\left(-v^{\star\prime}(\sigma)\right)^{q-p+1}d\sigma\right]d\tau,\quad s\in\left[0,|\Omega|\right]. \tag{3.12}$$

Next, we introduce the following key lemma to prove Theorem 1.2, the proof of which is motivated by [17,23].

Lemma 3.4. *If* (1.3) *holds, then the equation*

$$z=Kz \tag{3.13}$$

has no solution in $D_1(K)$.

Proof. We argue by contradiction. Assume that there exists a solution $z\in D_1(K)$ of (3.13).

Step 1. Let $z_0(s)=f_0 s$ and

$$z_m(s)=\int_0^s \theta\left(NC_N^{\frac{1}{N}}\tau^{1-\frac{1}{N}}\right)^{-\gamma}\underline{v}^{-\frac{\gamma}{p}}(\tau)z_{m-1}^{\gamma}(\tau)\,d\tau,\quad s\in\left[0,|\Omega|\right]. \tag{3.14}$$

We claim that

$$z(s)\geqslant\sum_{m=0}^{k} z_m(s),\quad \forall k\in N \tag{3.15}$$

and

$$z_m(s)\geqslant\theta^{\frac{\gamma^m-1}{\gamma-1}}\left(NC_N^{\frac{1}{N}}\right)^{-\frac{\gamma^{m+1}-\gamma}{\gamma-1}}(\gamma-1)^{\frac{\gamma^m-1}{\gamma-1}}\gamma^{\frac{m}{\gamma-1}-\frac{\gamma(\gamma^m-1)}{(\gamma-1)^2}}\times A\left(|\Omega|\right)^{-\frac{\gamma^m-1}{\gamma-1}}f_0^{\gamma^m}s^{-\frac{\gamma^m-\gamma}{\gamma-1}}F_1^{\frac{\gamma^m-1}{\gamma-1}}(s),\quad m\geqslant1. \tag{3.16}$$

In fact, we prove (3.15) by induction on k. Since $z(s)=Kz(s)\geqslant f_0s=z_0(s)$, we obtain

$$\begin{aligned}z(s)&=f_0s+\theta\int_0^s\left(NC_N^{\frac{1}{N}}\tau^{1-\frac{1}{N}}\right)^{-\gamma}\underline{v}^{-\frac{\gamma}{p}}z^{\gamma}(\tau)\,d\tau\\&\geqslant z_0(s)+z_1(s),\end{aligned}$$

which proves the claim for $k=1$. Now assume (3.15) holds for some $k\in N$ and let us show that it also holds for $k+1$. Indeed,

$$\begin{aligned}z(s)&\geqslant z_0(s)+\theta\int_0^s\left(NC_N^{\frac{1}{N}}\tau^{1-\frac{1}{N}}\right)^{-\gamma}\underline{v}^{-\frac{\gamma}{p}}\left(\sum_{m=0}^{k}z_m(\tau)\right)^{\gamma}d\tau\\&\geqslant z_0(s)+\theta\sum_{m=0}^{k}\int_0^s\left(NC_N^{\frac{1}{N}}\tau^{1-\frac{1}{N}}\right)^{-\gamma}\underline{v}^{-\frac{\gamma}{p}}z_m^{\gamma}(\tau)\,d\tau\\&=z_0(s)+\sum_{m=0}^{k}z_{m+1}(s)=\sum_{m=0}^{k+1}z_m(s).\end{aligned}$$

This proves (3.15).

To show (3.16), we first see

$$z_m(s)\geqslant\theta^{\frac{\gamma^m-1}{\gamma-1}}\left(NC_N^{\frac{1}{N}}\right)^{-\frac{\gamma^{m+1}-\gamma}{\gamma-1}}A\left(|\Omega|\right)^{-\frac{\gamma^m-1}{\gamma-1}}f_0^{\gamma^m}s^{-\frac{\gamma^m-\gamma}{\gamma-1}}F_m(s),\quad m\geqslant1, \tag{3.17}$$

where

$$F_{m+1}(s)=\int_0^s\tau^{\frac{\gamma}{N}}\underline{v}^{-\frac{\gamma}{p}}(\tau)A^{\gamma}(\tau)F_m^{\gamma}(\tau)\,d\tau. \tag{3.18}$$

In fact, by virtue of (3.14), we can easily verify the case $m=1$ for (3.17). Moreover, if (3.17) holds for some m, then

$$\begin{aligned}z_{m+1}(s) &= \int_0^s \theta\bigl(NC_N^{\frac{1}{N}}\tau^{1-\frac{1}{N}}\bigr)^{-\gamma}\underline{\nu}^{-\frac{\gamma}{p}}(\tau)z_m^{\gamma}(\tau)\,d\tau\\ &\geqslant \theta\bigl(NC_N^{\frac{1}{N}}s\bigr)^{-\gamma}\int_0^s \tau^{\frac{\gamma}{N}}\underline{\nu}^{-\frac{\gamma}{p}}(\tau)\theta^{\frac{\gamma^{m+1}-\gamma}{\gamma-1}}\bigl(NC_N^{\frac{1}{N}}\bigr)^{-\frac{\gamma^{m+2}-\gamma^2}{\gamma-1}}\times A\bigl(|\Omega|\bigr)^{-\frac{\gamma^{m+1}-\gamma^2}{\gamma-1}}f_0^{\gamma^{m+1}}\tau^{-\frac{\gamma^{m+2}-\gamma^2}{\gamma-1}}F_m^{\gamma}(\tau)\,d\tau\\ &= \theta^{\frac{\gamma^{m+1}-1}{\gamma-1}}\bigl(NC_N^{\frac{1}{N}}\bigr)^{-\frac{\gamma^{m+2}-\gamma}{\gamma-1}}A\bigl(|\Omega|\bigr)^{-\frac{\gamma^{m+1}-1}{\gamma-1}}f_0^{\gamma^{m+1}}s^{-\frac{\gamma^{m+1}-\gamma}{\gamma-1}}\times\int_0^s \tau^{\frac{\gamma}{N}}\underline{\nu}^{-\frac{\gamma}{p}}(\tau)A^{\gamma}(\tau)F_m^{\gamma}(\tau)\,d\tau.\end{aligned}$$

Consequently, (3.17) holds.

Now we prove

$$F_m(s)\geqslant\prod_{\delta=1}^{m}\biggl(\frac{\gamma-1}{\gamma^{\delta}-1}\biggr)^{\gamma^{m-\delta}}F_1^{\frac{\gamma^m-1}{\gamma-1}}(s),\quad m\geqslant 1. \tag{3.19}$$

The case $m=1$ is obvious. Suppose (3.19) holds for some m. Then

$$\begin{aligned}F_{m+1}(s) &= \int_0^s \tau^{\frac{\gamma}{N}}\underline{\nu}^{-\frac{\gamma}{p}}(\tau)A^{\gamma}(\tau)F_m^{\gamma}(\tau)\,d\tau\\ &\geqslant \prod_{\delta=1}^{m}\biggl(\frac{\gamma-1}{\gamma^{\delta}-1}\biggr)^{\gamma^{m+1-\delta}}\int_0^s \tau^{\frac{\gamma}{N}}\underline{\nu}^{-\frac{\gamma}{p}}(\tau)A^{\gamma}(\tau)F_1^{\frac{\gamma^{m+1}-\gamma}{\gamma-1}}(\tau)\,d\tau\\ &= \prod_{\delta=1}^{m}\biggl(\frac{\gamma-1}{\gamma^{\delta}-1}\biggr)^{\gamma^{m+1-\delta}}\int_0^s F_1^{\frac{\gamma^{m+1}-\gamma}{\gamma-1}}(\tau)\,dF_1(\tau)\\ &= \prod_{\delta=1}^{m+1}\biggl(\frac{\gamma-1}{\gamma^{\delta}-1}\biggr)^{\gamma^{m+1-\delta}}F_1^{\frac{\gamma^{m+1}-1}{\gamma-1}}(s),\end{aligned}$$

which gives (3.19).

Moreover, after some calculations, we find

$$\begin{aligned}\prod_{\delta=1}^{m}\biggl(\frac{\gamma-1}{\gamma^{\delta}-1}\biggr)^{\gamma^{m-\delta}} &\geqslant \prod_{\delta=1}^{m}\biggl(\frac{\gamma-1}{\gamma^{\delta}}\biggr)^{\gamma^{m-\delta}} = (\gamma-1)^{\sum_{\delta=1}^{m}\gamma^{m-\delta}}\gamma^{-\sum_{\delta=1}^{m}\delta\gamma^{m-\delta}}\\ &= (\gamma-1)^{\frac{\gamma^m-1}{\gamma-1}}\gamma^{\frac{m}{\gamma-1}-\frac{\gamma(\gamma^m-1)}{(\gamma-1)^2}}.\end{aligned} \tag{3.20}$$

Then combining (3.17), (3.19) and (3.20), we complete the proof of (3.16).

Step 2. We assert that there exists a constant $s^*\in[0,|\Omega|]$ such that $\sum_{m=0}^{\infty}z_m(s)=\infty$, for $s\in[s^*,|\Omega|]$.

In fact, in view of (3.16), we have

$$z_m(s)\geqslant E(s)\bigl(\theta^{\frac{1}{\gamma-1}}\bigl(NC_N^{\frac{1}{N}}\bigr)^{-\frac{\gamma}{\gamma-1}}(\gamma-1)^{\frac{1}{\gamma-1}}\gamma^{-\frac{\gamma}{(\gamma-1)^2}}A\bigl(|\Omega|\bigr)^{-\frac{1}{\gamma-1}}f_0s^{-\frac{\gamma}{\gamma-1}}F_1^{\frac{1}{\gamma-1}}(s)\bigr)^{\gamma^m}\gamma^{\frac{m}{\gamma-1}},$$

where $E(s)=\theta^{\frac{1}{1-\gamma}}(NC_N^{\frac{1}{N}})^{\frac{\gamma}{\gamma-1}}(\gamma-1)^{\frac{1}{1-\gamma}}\gamma^{\frac{\gamma}{(\gamma-1)^2}}A(|\Omega|)^{\frac{\gamma}{\gamma-1}}s^{\frac{\gamma}{\gamma-1}}F_1^{\frac{1}{1-\gamma}}(s)$.

Take

$$s^*=F_1^{-1}\biggl(\frac{\gamma^{\frac{\gamma}{\gamma-1}}(NC_N^{\frac{1}{N}})^{\gamma}|\Omega|^{\eta+\gamma}\|\nu\|_{L^h(\Omega)}\||x|^l\|_{L^r(\Omega)}}{\theta(\gamma-1)f_0^{\gamma-1}}\biggr),$$

where F_1^{-1} is the inverse function of F_1. By virtue of (1.3), it is easy to check that $s^*\leqslant|\Omega|$. Moreover, since

$$A\bigl(|\Omega|\bigr)=\int_{\Omega}\nu|x|^l\,dx\leqslant\|\nu\|_{L^h(\Omega)}\bigl\||x|^l\bigr\|_{L^r(\Omega)}|\Omega|^{\eta},$$

for $s \geqslant s^*$,

$$\theta^{\frac{1}{\gamma-1}}\left(NC_N^{\frac{1}{N}}\right)^{-\frac{\gamma}{\gamma-1}}(\gamma-1)^{\frac{1}{\gamma-1}}\gamma^{-\frac{\gamma}{(\gamma-1)^2}}A\left(|\Omega|\right)^{-\frac{1}{\gamma-1}}f_0 s^{-\frac{\gamma}{\gamma-1}}F_1^{\frac{1}{\gamma-1}}(s) \geqslant 1.$$

Then

$$z_m(s) \geqslant E(s)\gamma^{\frac{m}{\gamma-1}}, \quad s \in \left[s^*, |\Omega|\right].$$

Noting $\gamma > 1$, it follows $\lim_{m\to\infty}\gamma^{\frac{m}{\gamma-1}} = \infty$. Thus, $\sum_{m=0}^{\infty} z_m(s) = \infty$ for $s \in [s^*, |\Omega|]$.

However, recalling (3.15) and noting that $z \in D_1(K)$, we obtain

$$Ms \geqslant z(s) \geqslant \sum_{m=0}^{\infty} z_m(s) = \infty, \quad s \in \left[s^*, |\Omega|\right],$$

which is a contradiction. □

Proof of Theorem 1.2. We proceed by contradiction. Assume that there exists a spherically symmetric solution to problem (P2). From Remark 3.2, we see that w defined by (3.12) is a solution of (3.13) in $D_1(K)$. This contradicts with Lemma 3.4. □

Proof of Theorem 1.3. Let

$$\rho(s) = \int_0^s f^\star(\tau)\exp\left[\theta\int_\tau^s \left(NC_N^{\frac{1}{N}}\sigma^{1-\frac{1}{N}}\right)^{-(p-q)}\underline{\nu}^{\frac{q}{p}-1}(\sigma)\left(-u^{\star\prime}(\sigma)\right)^{q-p+1}d\sigma\right]d\tau.$$

Then we can get $\rho \in D_2(K)$ and $\rho(s) \leqslant K\rho(s)$ for $s \in [0, |\Omega|]$ in almost exactly the same way that we estimate $\widetilde{w}$ in Step 2 of the proof of Theorem 1.1.

On the other hand, since $w \in D_2(K)$ and $w(s) = Kw(s)$ for $s \in [0, |\Omega|]$, by recalling that K has property (m) in $D_2(K)$ and $R_2(K) \subseteq D_2(K)$, we get from Lemma 3.1 that

$$\rho(s) \leqslant w(s), \quad s \in \left[0, |\Omega|\right].$$

Moreover, (3.5) and (3.6) imply

$$\left(-u^{\star\prime}(s)\right)^{p-1}\left(NC_N^{\frac{1}{N}}s^{1-\frac{1}{N}}\right)^p\underline{\nu} \leqslant \rho(s), \quad \text{a.e. } s \in \left(0, |\Omega|\right)$$

and

$$\left(-v^{\star\prime}(s)\right)^{p-1}\left(NC_N^{\frac{1}{N}}s^{1-\frac{1}{N}}\right)^p\underline{\nu} = w(s), \quad \text{a.e. } s \in \left(0, |\Omega|\right).$$

Thus,

$$-u^{\star\prime}(s) \leqslant -v^{\star\prime}(s), \quad \text{a.e. } s \in \left(0, |\Omega|\right).$$

Observing $u^\star(|\Omega|) = v^\star(|\Omega|) = 0$, we have

$$u^\star(s) \leqslant v^\star(s), \quad s \in \left[0, |\Omega|\right].$$

Finally, (1.5) can be proved by proceeding as in [11]. Thus we complete the proof of Theorem 1.3. □

Proof of Theorem 1.4. Let $h > 0$ and $t \geqslant 0$. Taking

$$\psi_i(x) = \begin{cases} 1 & \text{if } u_i(x) > t+h, \\ \frac{u_i(x)-t}{h} & \text{if } t < u_i(x) \leqslant t+h, \\ 0 & \text{otherwise} \end{cases}$$

in (1.6), and then letting $h \to 0$, we have

$$\begin{aligned} -\frac{d}{dt}\int_{\{u_1>t\}} \nu|\nabla u_1|^p\,dx &\leqslant -\frac{d}{dt}\int_{\{u_1>t\}} a(x,u,\nabla u_1)\cdot\nabla u_1\,dx \\ &\leqslant \int_{\{u_1>t\}} \left(f_1 + \theta_1\nu^{\frac{q}{p}}|\nabla u_1|^q\right)dx \end{aligned}$$

and

$$\frac{d}{dt}\int_{\{u_2>t\}} \nu|\nabla u_2|^p\,dx \geqslant -\frac{d}{dt}\int_{\{u_2>t\}} a(x,u,-\nabla u_2)\cdot\nabla u_2\,dx \geqslant \int_{\{u_2>t\}} \left(-f_2-\theta_2\nu^{\frac{q}{p}}|\nabla u_2|^q\right)dx.$$

That is

$$-\frac{d}{dt}\int_{\{u_i>t\}} \nu|\nabla u_i|^p\,dx \leqslant -\frac{d}{dt}\int_{\{u_i>t\}} a(x,u,\nabla u_i)\cdot\nabla u_i\,dx \leqslant \int_{\{u_i>t\}} \left(f_i+\theta_i\nu^{\frac{q}{p}}|\nabla u_i|^q\right)dx, \quad i=1,2.$$

In view of $|u_i>0|\leqslant\frac{|\Omega|}{2}$ (see [18]), it follows

$$\min{}^{1-\frac{1}{N}}\left\{\left|\mu_i(t)\right|,\left|\Omega\backslash\mu_i(t)\right|\right\}=\left|\mu_i(t)\right|^{1-\frac{1}{N}}, \quad i=1,2.$$

Thus by the relative isoperimetric inequality (2.1) and Hölder inequality, we have

$$1\leqslant Q\,\mu_i(t)^{\frac{1}{N}-1}\left(-\frac{d}{dt}\int_{\{u_i>t\}}\nu|\nabla u_i|^p\,dx\right)^{\frac{1}{p}}\underline{\nu_i}^{-\frac{1}{p}}\left(\mu_i(t)\right)\left(-\mu_i'(t)\right)^{1-\frac{1}{p}}, \quad i=1,2. \tag{3.21}$$

Now we can proceed by using (3.21), Gronwall inequality and the properties of rearrangement to get

$$\left(-u_i^{\star'}(s)\right)^{p-1}\left(Q^{-1}s^{1-\frac{1}{N}}\right)^p\underline{\nu_i}(s) \leqslant \int_0^s f_i^\star(\tau)\exp\left[\theta_i\int_\tau^s\left(Q^{-1}\sigma^{1-\frac{1}{N}}\right)^{q-p}\underline{\nu_i}^{\frac{q}{p}-1}(\sigma)\left(-u_i^{\star'}(\sigma)\right)^{q-p+1}d\sigma\right]d\tau, \quad \text{a.e. } s\in\left(0,\frac{|\Omega|}{2}\right).$$

Furthermore, the assumption (1.7) ensures that problem (P4i) admits a unique spherically symmetric solution v_i. This can be done by following the proofs of Theorem 1.1, where the operator K is replaced by

$$K_i\psi(s)=\int_0^s f_i^\star(\tau)+\theta_i\left(Q^{-1}\tau^{1-\frac{1}{N}}\right)^{-\gamma}\underline{\nu}_i^{-\frac{\gamma}{p}}(\tau)\psi^\gamma(\tau)\,d\tau, \quad s\in\left[0,\frac{|\Omega|}{2}\right],\ i=1,2.$$

Then v_i satisfies

$$\left(-v_i^{\star'}(s)\right)^{p-1}\left(Q^{-1}s^{1-\frac{1}{N}}\right)^p\underline{\nu_i}(s) = \int_0^s f_i^\star(\tau)\exp\left[\theta_i\int_\tau^s\left(Q^{-1}\sigma^{1-\frac{1}{N}}\right)^{q-p}\underline{\nu_i}^{\frac{q}{p}-1}(\sigma)\left(-v_i^{\star'}(\sigma)\right)^{q-p+1}d\sigma\right]d\tau, \quad \text{a.e. } s\in\left(0,\frac{|\Omega|}{2}\right).$$

Thus, we can follow the proof of Theorem 1.3 to get

$$-u_i^{\star'}(s)\leqslant -v_i^{\star'}(s), \quad s\in\left(0,\frac{|\Omega|}{2}\right),\ i=1,2. \tag{3.22}$$

Since $u_i^\star\in C(0,\frac{|\Omega|}{2}]$ and $|u_i>0|\leqslant\frac{|\Omega|}{2}$, we have $u_i^\star(\frac{|\Omega|}{2})=0$. Integrating (3.22) from s to $\frac{|\Omega|}{2}$, we complete the proof of (1.8).

On the other hand, (1.9) can be proved by proceeding as in [11]. □

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